Recently, statistical fluctuations are revealing interesting features for a number of one-dimensional systems confined to circular geometries. For instance, for semiflexible polymers of a fixed length, such as constrained DNA rings, the closure condition influences the scaling, shape, and transport behavior [1]. Topology is actually expected to play a key role in a large number of related biophysical processes, such as, e.g., translocation in nanochannels or nanopores [2,3], or knot localization [4]. In particular, circular DNA molecules in two dimensions have been experimentally found [1] to be well described as pressurized vesicles [5], their scaling behavior depending on the geometry [6]. Thus, for deflated rings (negative pressure difference $\Delta p$), fluctuations are in the universality class of lattice animals, while for $\Delta p = 0$, statistics are those of a ring self-avoiding walk (SAW) [1]. The latter is important as a paradigmatic model of polymers [7] and because the SAW is believed to constitute a conformally invariant system in two dimensions [8].

For planar rings evolving far from equilibrium, the closure condition is also proving to be nontrivial, as recently observed for the edge of a drying colloidal suspension [10,11], and for many more systems, from epitaxy to bacterial growth [12]. Thus, as proposed in Ref. [13], the probability distribution function (PDF) of the height fluctuations for interfaces which, as many of these, belong to the Kardar-Parisi-Zhang (KPZ) universality class [14,15], depends on the global curvature. The eponymous equation [16], which is the prime representative for these systems, is a continuum model for the evolution of a rough interface between a (stable, e.g., solid) phase that grows at the expense of a (an unstable, e.g., vapor) phase,

$$
\partial_t h = v + \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x,t),
$$

(1)

where $h(x,t)$ is the height field above the substrate position $x \in \mathbb{R}^d$ at time $t$, $\eta$ is Gaussian white noise, $v$ is the growth speed for a flat interface, and $\nu > 0$, $\lambda$ are additional parameters. On the one hand, from the point of view of the theory of stochastic processes, the KPZ equation features a remarkable example of a time crossover [17,18] between the two main universality classes of kinetic roughening [14,15], namely, the Edwards-Wilkinson (EW) class at short times, and the KPZ class at long times. Experimentally [9], however, while such a crossover may have been seen in $d = 1$ for interfaces with a null global curvature, it has not for the circular geometry case. On the other hand, for such ring-shaped interfaces, height statistics are indeed distinctively described [13] by the Tracy-Widom (TW) PDF for the largest eigenvalue of large random matrices in the Gaussian unitary ensemble (GUE), as recently supported by exact solutions of the KPZ equation on an infinite substrate and a wedge initial condition (but without explicit closure), or for related systems [19]. This actually occurs with a remarkable degree of universality [17,20], as the same PDF, critical exponents, and limiting processes apply to discrete models, continuum equations [21], and experiments [9–11]. Hence, in two dimensions (2D) both pressurized vesicles and the KPZ equation notably demonstrate the nontrivial role of geometry, as a part of the universality class and related renormalization-group fixed point [6], in and out of equilibrium, respectively.

Note, Eq. (1) is just the small-slope, single-valued approximation of a more general equation [22],

$$
\partial_t \mathbf{r} = [A_0 + A_1 K(\mathbf{r}) + A_2 \eta(\mathbf{r},t)] \mathbf{u}_n,
$$

(2)

where $\mathbf{r}(t) \in \mathbb{R}^{d+1}$ gives the interface position, $K(\mathbf{r})$ is the local extrinsic surface curvature, $\mathbf{u}_n$ is the normal direction pointing towards the unstable phase, and constants $A_0$, $A_1$, and $A_2$ relate to parameters in Eq. (1) in a simple way [22]. They account for, respectively, the average growth speed along the local normal direction, surface tension effects, and noise in the local growth velocity, precisely the physical mechanisms at play in the formulation of the KPZ equation as a continuum interface model [16]. However, those produced by Eq. (2) are not constrained to small slopes or lack of overhangs [22].

For a ring geometry and $d = 1$, Eq. (1) actually has to be given up in favor of Eq. (2), since the closure condition hinders description of the interface profile by a single-valued function altogether. Alternative formulations to Eq. (2) are available (see, e.g., in Ref. [12]), although most include the neglect of overhangs and/or additional simplifications. A natural question is then whether Eqs. (1) and (2) have the same
Curves for times \( t \) for \( R_0 = 10, A_0 = 0.01, A_1 = 0.01 \), and \( A_n = 1 \), with \( \Delta t = 0.1 \), \( l_{\text{min}} = 0.1 \), and \( l_{\text{max}} = 1 \). Curves for times \( t = 2, 20, 150, 1000, \) and 5000, inner to outer. (b) Rescaled view to ease comparison. All units are arbitrary.
I, while the average radius remains almost constant, this length increases due to fluctuations. In regime II, when \( R(t) \) grows steadily, \( L(t) \) actually becomes proportional to it. This behavior is appreciated in Fig. 2(b), where the \( L(t) \) is plotted versus \( R(t) \). There is a threshold total length, proportional to \( R_0 \), below which no radial growth occurs, and above which both measures become proportional. As seen in Fig. 1, prior to regime II, noise basically induces loss of the initial circular symmetry. In Fig. 2(c) we quantify this effect by plotting the asymmetry parameter \([26]\) \( \Sigma(t) = (S_{21}^2/S_{22}^2) \), i.e., the ratio of the smallest to largest eigenvalues, \( S_{21}, S_{22} \), of the gyration tensor \( S \). This is frequently used to assess polymer classes in terms of self-avoidance, dimensionality, rigidity, etc \([1,6,27,28]\). In our case, \( \Sigma(t) \) decreases with time, reaches a minimum value \( \Sigma(t \approx 400) \approx 0.68 \), and increases back, approaching the characteristic value of a circular swollen polymer in regime II. Other measures of anisotropy lead to the same conclusion [see, e.g., Fig. 2(c) for the quadrupole moment, \( Q^2 = \langle (x^2 - y^2)^2 \rangle / \langle r^2 \rangle \), where \( x, y, \) and \( r \) are relative to the center of mass (CM) of the \( \{ P_i \}_{i=1}^{N} \) point distribution].

Given the relevance of fluctuations in these dynamics, we assess them in Fig. 3(a), where we show the time evolution of the global interface roughness, \( W_{CM}(t) = \langle P_i(t) - R_{CM}(t) \rangle_{i=1}^{N}^{1/2} \), with \( R_{CM}(t) \) being the position of the CM \([29]\). Scaling behavior \( W(t) \sim t^{\beta} \) holds, with \( \beta \approx 1/3 \), both in regimes I and II. As standard for kinetic roughening systems in a circular geometry, \( W(t) \) does not saturate due to the uninterrupted growth of the system size \([30]\). Moreover, as indicated in Fig. 2(c), during regime II the quadrupole moment \( Q \) decays as \( t^{-2/3} \), which follows if we estimate \( Q \) as the ratio of the radial fluctuations to the average radius, \( t^{\theta=1} \sim t^{-2/3} \). We also measure the local roughness \( w(l,t) \), namely, the interface fluctuations (restricted to windows of size \( l \)) around a fitting circular arc, which is drawn with respect to the CM. Data are shown in Fig. 3(c) as functions of \( l \), for several times. Scaling behavior ensues, \( w(l) \sim l^\alpha \), provided that, as in the standard Family-Vicsek (FV) ansatz \([14]\), the window size \( l \) is smaller than a correlation length \( \xi(t) \), which itself grows as \( \xi(t) \sim t^{1/2} \). The FV scaling relation \( z = \alpha/\beta \) holds for exponent values \( (\alpha,z) \) which are \((2/3,2)\) in regime I, and \((1/2,3/2)\) in regime II [see Fig. 3(d)]. Indeed, in both cases \( \beta = 1/3 \), as implied by \( W(t) \). Hence, the fractal dimension \( D_F = 2 - \alpha \) \([14]\) changes from 4/3 in regime I to 3/2 in regime II. Overall, the evolution is from kinetic roughening.
roughening in the SAW class (regime I), for which $\alpha_{\text{SAW}} = 2/3$ [7] and $\beta_{\text{SAW}} = 1/3$ [31,32], to asymptotic KPZ scaling in regime II, for which $\alpha_{\text{KPZ}} = 1/2$ and $\beta_{\text{KPZ}} = 1/3$ [14]. If the noise amplitude decreases significantly ($A_n \lesssim 0.01$), the roughness remains constant in regime I, namely, the SAW stage disappears, the only measurable scaling behavior being the KPZ asymptotics in regime II, as in the experiments for circular geometry [9]. See Fig. 3(b).

The progressive dominance of radial fluctuations can be appreciated in Fig. 3(a), where we plot the time evolution for the standard deviation of the random variable $R(t)$. Although this quantity grows fast with $t$, numerically it remains much smaller than $W(t)$ until the onset of regime II, after which both remain proportional. Actually, we can further explore radial fluctuations in the asymptotic KPZ regime. Thus, following Prahofer and Spohn [13], we rewrite

$$R(t) \approx \rho_0 + V t + \Gamma t^\beta \chi,$$

where $\rho_0$, $V$, and $\Gamma$ are constants, $\beta = \beta_{\text{KPZ}}$, and $\chi$ is a random variable with zero average and unit variance, whose probability distribution is stationary and corresponds to the TW-GUE distribution [9–11,13,19]. We have collected the instantaneous data for 17 different times in the range $t \in [700,1500]$, i.e., well within regime II, for 2500 noise realizations, in order to check this conjecture. Results are shown in Fig. 4(a), where we plot the probability distribution of $\chi$, obtained following the procedure described in Ref. [33], and compare it with the analytical result [17]. Moreover, we have measured the third and fourth cumulants of this $\chi$ distribution, i.e., the skewness and kurtosis, which are, respectively, $\langle \chi^3 \rangle_c \approx 0.233$ and $\langle \chi^4 \rangle_c \approx 0.733$. The theoretical values for the TW-GUE distribution are $13.0224$ for the skewness and $0.093$ for the kurtosis, which are close enough. For comparison [13], for the TW-GUE distribution, the skewness is $0.293$ and the kurtosis $0.165$, both being zero for the Gaussian distribution. Figure 4(b) shows the time evolution of the cumulants of $R(t)$ towards the TW-GUE values. We must remark on the negative sign that we obtain for parameter $\Gamma$ in Eq. (4), implying a negative skewness for $R(t)$. Physically, this is due to the fact (data not shown) that, in regime II, the number of cavities removed per unit length and unit time by the self-intersection removal condition is smaller than the number of removed outgrowths.

In summary, while for relatively small noise, perhaps as the experimentally studied case [9], only KPZ scaling is obtained, for large-noise intensities Eq. (2) predicts a circular interface to cross over in time between an early-time SAW regime, in which it behaves as a freely fluctuating ring “polymer,” and the late-time regime controlled by KPZ fluctuations in the presence of nonzero average curvature. For small times, the local driving does not suffice to counteract fluctuations, so that the average circular shape smears out, interactions among interface points being controlled by surface tension (note the dynamic exponent indeed is $z = 2$ in regime I). Since $\xi(t)$ increases while $R(t)$ stays almost constant, eventually the system becomes fully correlated. From that time on, the increasing length needs to be accommodated in the finite area enclosed by the initial radius, and the intersection removal mechanism becomes relevant, smoothing out the interface. Because of the average (convex) circular geometry, cavities are removed more frequently than outgrowths, and the interface starts to grow, leading to the expected KPZ regime, with TW-GUE characteristics.

Our results conspicuously connect the celebrated 2D SAW and KPZ universality classes, both of which underscore the importance of geometrical constraints for scaling behavior, in and out of equilibrium. Crucially, the transition in time between them can only be elucidated through the existence of overhangs, which eludes other continuum models of kinetic roughening. Hence, the large-noise regime I of Eq. (2) might constitute a scaling limit for the 2D ring SAW [8], while providing an efficient algorithmic procedure to generate them [34]. Alternative connections between the KPZ and SAW classes are available, namely, between isoheight lines of the 2+1 dimensional (3D) KPZ equation and 2D SAW-related formulations [35]. In general, the conformation and dynamics of circular polymers are still subjects of considerable interest [36]. Current experimental capabilities reach even down to single molecule experiments [1], so that one might speculate on the possibility to observe a dynamical transition of the type elucidated here in appropriate nonequilibrium, 2D constrained settings [37].

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CIRCULAR KARDAR-PARISI-ZHANG EQUATION AS AN . . . PHYSICAL REVIEW E 89, 010401(R) (2014)

[25] See Supplemental Material at http://link.aps.org supplemental/10.1103/PhysRevE.89.010401 for further details on Eqs. (2) and (3), and the simulation and measurement procedures that we have employed.
[29] We have verified that—in contrast with cases such as the Eden model [see S. C. Ferreira, Jr. and S. G. Alves, J. Stat. Mech. (2006) P11007]—measurements of the roughness with respect to a fixed point instead of the CM do not change results on critical exponent values.
[33] To obtain the histogram for $\chi$, we follow a procedure similar to Ref. [9]. Namely, the raw $(t, R)$ pairs are fit to a linear function of time, providing $\rho_0$ and $V$. The squared residuals are fit to a power law of time, yielding $\Gamma_1$ and $\beta \approx 0.3353$. From here we estimate the values $\chi = (R - \rho_0 - V t)/\Gamma_1 t^\beta$. Finally, $\Gamma_1$ has the same sign as the skewness of $R(t)$.